



TITLE:

On a theta integral (Automorphic Forms and L -Functions)

AUTHOR(S):

Ninomiya, Hirohito

CITATION:

Ninomiya, Hirohito. On a theta integral (Automorphic Forms and L -Functions). 数理解析研究所講究録 1999, 1103: 39-45

ISSUE DATE:

1999-06

URL:

<http://hdl.handle.net/2433/63204>

RIGHT:

On a theta integral

Hirohito Ninomiya(Kyushu University)

1 Introduction.

Around the begining of 70's, Doi-Naganuma and Shimura discovered correspondences between certain spaces of modular forms, being compatible with the Hecke operators [Sh, DN]. Shimura's correspondence was investigated by Shintani and Niwa, using the Weil representations [Sn, Nw].

The construction of holomorphic cusp forms by Niwa was generalized by many authers [Kd, Za, Od, RS, Kj], namely Oda and Rallis-Shiffman independently considered the case of orthogonal groups $O(2, N)$ for general N . (Zagier constructed the holomorphic kernel function for Doi-Naganuma's correspondence.)

Recently, Borchers discovered a family of automorphic forms with infinite product on orthogonal groups of signature $(2, N)$. At the same time, Physicists [AFGNT, HM] started to study and developed a variant of theta correspondence from another direction. Especially, Harvey and Moore found certain theta integral express the automorphic form of Borchers [HM, Bo2].

The purpose of this note is to give a construction of meromorphic automorphic forms on orthogonal groups of the signature $(2, N)$, $N \geq 2$, using the same kind of theta correspondence, (in section 4,) without proofs. This generalizes a result of Antoniadis, Ferrara, Gava, Narain and Taylor [AFGNT], which dealt with the case $N = 1, 2$.

We note that, in the classical (positive weight) case, Maass' lifting [Ma, Gr, Su] and theta correspondence are known to be coincide up to non-zero scalar multiplication, (at least S is maximal even, as far as I know.) In the negative weight case, the situation is different.

The following facts for $SL_2(\mathbf{R})$ are well known ($k \in \mathbf{Z}$, $k \geq 2$, $D_\tau = \frac{\partial}{\partial \tau}$, and \mathcal{H}_1 is the upper half plane):

(1.1) for $f \in C^\infty(\mathcal{H}_1)$ and $g \in SL_2(\mathbf{R})$,

$$D_\tau^{k-1}(f|_{2-k}g) = (D_\tau^{k-1}f)|_kg.$$

(1.2) A holomorphic function f on \mathcal{H}_1 satisfies $D_\tau^{k-1}f = 0$ if and only if f is a polynomial in τ of degree at most $k - 2$.

(1.3) For a holomorphic function F on \mathcal{H}_1 , set

$$f = \int_{\tau_0}^{\tau} F(\tau') \frac{(\tau - \tau')^{k-2}}{(k-2)!} d\tau'.$$

Then it satisfies $D_{\tau}^{k-1} f = F$.

In section 3, we will find analogous statements in the case of orthogonal groups $O(2, N)$. Then, by integrating the form constructed in section 4, one can obtain a function, which behaves like an automorphic form of negative weight, but is multi-valued, and analytic function with logarithmic singularities. This form is obtained from Maass' lifting. The same construction for $N = 2$ already appears in [AFGNT] (See also [FS]). Note that, [HM] and [Kw] also considered the case $N > 2$, but take slightly different construction.

2 Basic definitions.

Let $N \geq 2$, and S be an even integral symmetric matrix of degree $N + 2$ with signature $(2, N)$, and of the following form:

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (1)$$

The real orthogonal group $O(S)$ of S acts on the domain $H^* = \{z \in \mathbb{C}^N \mid \eta(z) := 2 {}^t \text{Im} z S_0 \text{Im} z > 0\}$ as follows: If we set

$$p(z) = {}^t \begin{pmatrix} -{}^t z S_0 z / 2 & {}^t z & 1 \end{pmatrix} \quad (z \in H^*), \quad (2)$$

then for any $g \in O(S)$ and $z \in H^*$, there exists unique $gz \in H^*$ and $\mu(g, z) \in \mathbb{C}^\times$ satisfying

$$g p(z) = p(gz) \mu(g, z).$$

Take one of the two connected components $H \subset H^*$, and we denote by $O(S)^+$ the set of the elements in $O(S)$, fixing H . Then $O(S)^+$ holomorphically acts on H . The action is transitive and the stabilizer of a point is isomorphic to $SO(2) \times O(N)$.

Write $D_z = {}^t \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_N} \right)$ for $z = {}^t (z_1, \dots, z_N) \in H$. Following Shimura [Sh1], we define

$$q(z) = {}^t (\eta(z) D_z (\eta(z)^{-1} {}^t p(z))). \quad (z \in H) \quad (3)$$

Then for any $g \in O(S)^+$ and $z \in H$, there exists $\lambda(g, z) \in O(S_0)_{\mathbb{C}}$ which satisfies the following:

$$g q(z) = q(gz) \lambda(g, z), \quad (4)$$

$$\begin{aligned} D_z(f \circ g) &= {}^t \lambda(g, z) ((D_z f) \circ g) \mu(g, z)^{-1} \\ (f \in C^\infty(H), g \in O(S)^+, z \in H), \end{aligned} \quad (5)$$

and we find that, $\mu(g, z)$ and $\lambda(g, z)$ are the holomorphic automorphic factor.

3 Differential calculus.

Set $V = \mathbb{C}^N$ and $V_0 = \{a \in \mathbb{C}^N \mid {}^t a S_0 a = 0\}$. The symmetric algebra $S(V) = \bigoplus_{l=0}^{\infty} S(V)_l$ of V possess a bilinear form \langle, \rangle satisfying $\langle a^l, b^l \rangle = ({}^t a b)^l$. We denote by $H(V)_l$ the subspace in $S(V)_l$ generated by a^l ($a \in V_0$) over \mathbb{C} . Then the representation of $O(S_0)_{\mathbb{C}}$ on $H(V)_l$ is irreducible, equivalent to the one on the space of harmonic polynomial of degree l , under the isomorphism between $S(V)$ and the ring of polynomial map on V . The latter is identified with the symmetric algebra $S(V^*)$ of the dual space of V . We have an isomorphism

$$H(V)_l = S(V)_l / Q S(V)_{l-2} \quad (l \geq 0),$$

where Q is the element in $S(V)_2$ corresponding to ${}^t x S_0 x \in S(V^*)_2$. We denote by π_l the projection of $S(V)_l$ on $H(V)_l$.

Proposition 3.1. Assume $r > 0$.

(1) For $f \in C^\infty(H)$, $g \in O(S)^+$ and $a \in V_0$,

$$\langle a^r, D_z^r (\mu(g, z)^{r-1} f(gz)) \rangle = \mu(g, z)^{-1} \langle \lambda(g, z) a^r, (D_z^r f)(gz) \rangle \quad (6)$$

(2) A holomorphic function $f \in C^\infty(H)$ satisfies $({}^t a D_z)^r f(z) = 0$ for all $a \in V_0$, if and only if f is a polynomial in z_1, \dots, z_N , ${}^t z S_0 z$ of degree at most $r - 1$ (as a polynomial in $N + 1$ variable.)

(3) Suppose $\Phi : H \rightarrow \text{Hom}_{\mathbb{C}}(H(V)_r \bigoplus_{l=0}^r S(V)_l, \mathbb{C})$ is a holomorphic function satisfying the following condition:

$${}^t b D_z \Phi(z, h) = \Phi(z, b h) \quad (b \in V_0, h \in H(V)_r \bigoplus_{l=0}^{r-1} S(V)_l). \quad (7)$$

Then there exists a holomorphic function f on H , such that

$$({}^t a D_z)^r f(z) = \Phi(z, a^r) \quad (a \in V_0). \quad (8)$$

In fact, if we set

$$\omega(z, z') = \sum_{l=0}^{r-1} b_l \frac{(Q/2 {}^t(z - z') S_0(z - z')/2)^l}{l!} \pi_{r-l} \left(\frac{(z - z')^{r-l-1}}{(r-l-1)!} dz' \right), \quad (9)$$

$$b_l = \prod_{j=1}^l (r - j + N/2 - 1)^{-1} \quad (r > l \geq 0) \quad (10)$$

then ω satisfies $d\omega = -dz' \wedge \omega$, $\Phi(z', \omega)$ is a closed 1-form, and the function

$$f(z) = \int_{z_0}^z \Phi(z', \omega(z, z')). \quad (11)$$

hold the required properties.

4 The construction of meromorphic automorphic forms.

Hereafter, we assume that S is unimodular, for simplicity (so it follows $N \equiv 2 \pmod{8}$). Set $M = \mathbf{Z}^{N+2} = {}^t(\mathbf{Z} \quad L \quad \mathbf{Z})$, and $O(M)^+ = O(S)^+ \cap GL_{N+2}(\mathbf{Z})$.

Define the Siegel theta function by

$$\theta_M(\tau, z, a^r) = \sum_{\lambda \in M} {}^t \lambda S \frac{p(z)}{\eta(z)} ({}^t \lambda S \overline{q(z)} a)^r \mathbf{e} \left(\bar{\tau} {}^t \lambda S \lambda / 2 + 2iy \frac{|{}^t \lambda S p(z)|^2}{\eta(z)} \right) \quad (12)$$

for $\tau = x + iy \in \mathcal{H}_1$, $z \in H$, and $a \in V_0$ ($r > 0$). Here, we denote by \mathcal{H}_1 the upper half plane.

Let $C(\tau)$ be a modular form of weight $k = 2 - r - N/2$ for $SL_2(\mathbf{Z})$, holomorphic on \mathcal{H}_1 , and meromorphic at the cusp $i\infty$. The Fourier expansion at $i\infty$ is given by

$$C(\tau) = \sum_{n \in \mathbf{Z}, n \geq -N_0} c(n) \mathbf{e}(n\tau)$$

for some $c(n) \in \mathbf{C}$, and $N_0 \geq 0$. We set $F = \{\tau \in \mathcal{H}_1 \mid |\tau| \geq 1, |Re(\tau)| \leq 1/2\}$, and $F_w = \{\tau \in F \mid Im(\tau) \geq w\}$ ($w \geq 1$).

Further we assume that $r > 0$ is odd, and $c(0) = 0$ in case $r = 1$.

Theorem 4.1.

1. The integral

$$\Phi(z, a^r) = \lim_{w \rightarrow \infty} \int_{F_w} \frac{dx dy}{y^2} y^2 C(\tau) \overline{\theta_M(\tau, z, \bar{a}^r)} \quad (z \in H, a \in V_0) \quad (13)$$

converges outside the quadratic divisors, and defines a meromorphic function on H , satisfying

$$\Phi(z, a^r) = \mu(g, z)^{-1} \Phi(gz, \lambda(g, z)a^r) \quad (14)$$

for all $g \in O(M)^+$.

2. For any compactly supported open set U in H , the singularities of Φ on U are given by the finite sum

$$\frac{1}{4\pi} \sum_{\lambda \in M, {}^t\lambda S \lambda < 0, U \cap H_\lambda \neq \emptyset} c({}^t\lambda S \lambda / 2) ({}^t a D_z^t \lambda S p(z))^r / {}^t \lambda S p(z) \quad (15)$$

where $H_\lambda = \{z \in H \mid {}^t \lambda S p(z) = 0\}$. (Precise meaning of the word “singularity” is that, the difference of two functions is extended to the C^∞ function on U .) Note that the inner expression of the sum can be rewritten as

$$({}^t a D_z)^r \left\{ \frac{({}^t \lambda S p(z))^{r-1}}{(r-1)!} \log({}^t \lambda S p(z)) \right\}.$$

3. Set $C = H \cap \mathbf{R}^N$, and take a connected component W_L of

$$C - \bigcup_{\mu \in L, {}^t \mu S_0 \mu < 0, c({}^t \mu S_0 \mu / 2) \neq 0} \{v \in C \mid {}^t \mu S_0 v = 0\}.$$

The Fourier expansion of Φ is given by

$$\Phi = -i \left(\frac{{}^t a D_z}{2\pi i} \right)^r \left\{ h(z) + \sum_{\mu \in L, {}^t \mu S_0 \mu > 0} c({}^t \mu S_0 \mu / 2) \sum_{n > 0} n^{-r} e(n {}^t \mu S_0 z) \right\} \quad (16)$$

for sufficiently large $\eta(z) > 0$, $Im z \in W_L$, and a harmonic polynomial $h(z)$ of degree r .

Example 4.2.[E,AFGNT]: Let $N = 2$, $S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $r = 3$, $k = 2 - 3 - 2/2 = -2$, $C(\tau) = E_{10}(\tau)/\Delta(\tau)$. Then

$$\Phi\left(\begin{pmatrix} \sigma \\ \tau \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}^3\right) = i \frac{j'(\sigma)}{j(\sigma) - j(\tau)} \frac{C(\tau)}{C(\sigma)}. \quad (17)$$

References

- [AFGNT] I.Antoniadis,S.Ferrara,E.Gava,K.S.Narain and T.R.Taylor, Perturbative prepotential and monodromies in N=2 heterotic superstring, Nucl.Phys.B447(1995)
- [Bo1] R.E.Borcherds, Automorphic forms on $O_{s+2,2}(\mathbf{R})$ and infinite products, Invent.Math.120,161-213(1995)
- [Bo2] R.Borcherds, Automorphic forms with singularities on Grassmannians, Invent.math.132,(1998)
- [FS] K.Förger and S.Stieberger,String Amplitudes and N=2,d=4 Prepotential in Heterotic $K3 \times T^2$ Compactifications,Nucl.Phys. B 514(1998)135, hep-th/9709004
- [HM] J.A.Harvey and G.Moore, Algebras,BPS states and strings, Nucl.Phys. B463,315(1996)
- [Kw] T.Kawai,String Duality and Enumeration of curves by Jacobi forms, hep-th/9804014.
- [Sh1] G.Shimura,The arithmetic of certain zeta functions and automorphic forms on orthogonal groups,Ann.of Math. 111(1980),313-375
- [DN] K.Doi and H.Naganuma, On the functional equation of certain Dirichlet series,Invent.math.9,1-14(1969)
- [Gr] V.G.Grithenko, Jacobi functions of n variables, Zap.Nauch. Sem. LOMI,168,32-45(1988)translated in J.Soviet Math.53,243-252(1991).
- [Kj] H.Kojima, On Construction of Siegel modular forms of degree two, J.Math. soc.Japan 34,393-411(1982)
- [Kd] S.Kudla.Theta-function and Hilbert modular forms,Nagoya Math. J.69,97-106(1978)

- [Ma] H.Maass,Über eine spezielschar von Modulformen Zweiten Grades 1,2,3, Inv.Math.52,95-104(1979), Inv.Math.53,249-253(1979), and Inv. Math.53,255-265(1979).
- [Nw] S.Niwa,Modular forms of half integral weight and the integral of certain theta-functions,Nagoya.Math.J.56,147-161(1974)
- [Od] T.Oda,,On modular forms associated with indefinite quadratic forms of signature $(2, n - 2)$,Math.Ann.231,97-144(1977)
- [RS] S.Rallis,G.Shiffman, On a relation between SL_2 cusp forms and cusp forms on the tube domain associated to orthogonal groups, Trans. Amer.Math.Soc.263,1-58(1981)
- [Sh] G.Shimura, On modular forms of half integral weight, Ann.of Math. 97,440-481(1973)
- [Sn] T.Shintani,On construction of holomorphic cusp forms of half integral weight,Nagoya.Math.J.58,83-126(1975)
- [Su] T.Sugano, Jacobi forms and theta lifting, Comment.Math.Univ. Sancti Pauli 44-1, 1-58(1995)
- [Za] D.Zagier, Modular forms associated to real quadratic field, Invent.math. 30,1-46(1975)

Graduate School of Mathematics,
 Kyushu University 33,Fukuoka 812 Japan
 e-mail adress: ninomiya@math.kyushu-u.ac.jp